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Computing Topological Entropy in a Space of Quartic Polynomials

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Abstract This paper adds a computational approach to a previous theoretical result illustrating how the complexity of a simple dynamical system evolves under deformations. The algorithm targets topological entropy in the 2-dimensional family P^Q of compositions of two logistic maps. Estimation of the topological entropy is made possible by the correspondence between P^Q and a subfamily of sawtooth maps P^T, and is based on the well-known fact that the kneading-data of a map determines its entropy. A complex search for kneading-data in P^T turns out to be computationally fast and reliable, delivering good entropy estimates. Finally, the algorithm is used to produce a picture of the entropy level-sets in P^Q, as illustration to theoretical results such as Hu (Ph.D. thesis, CUNY, 1995) and Radulescu (Discrete Cont. Dyn. Syst. 19(1):139–175, 2007).

Keywords Entropy · Computation · Kneading data · Isentropes

1 Preliminaries

1.1 m-Modal Maps

Let $h: I \to I$ be an *m-modal map* of the interval, i.e. there exist $0 < \mathbf{c}_1 \le \mathbf{c}_2 \le \cdots \le \mathbf{c}_m < 1$ *folding points* or *critical points* of h such that h is alternately increasing and decreasing on the intervals $H_0, \ldots, H_m$ between the folding points.

$$I = \bigcup_{k=0}^{m} H_k \cup \bigcup_{j=1}^{m} \{\mathbf{c}_j\}.$$

We say that h is of shape $s = (+, -, +, \ldots)$ if h is increasing on H_0 and of shape $s = (-, +, -, \ldots)$ if h is decreasing on H_0.

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If there is no smaller m with the properties above, then we call $l(h) = m + 1$ the lap number of h. We say h is *boundary anchored* if the boundary of the unit interval is invariant under h: $h(\{0, 1\}) = \{0, 1\}$.

We consider a simple order on the alphabet $\mathcal{A} = \{H_0, \ldots, H_m\} \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$, given by: $H_0 < \mathbf{c}_1 < H_1 < \cdots < \mathbf{c}_m < H_m$. We also define the *signature* sign A of any $A \in \mathcal{A}$ to be $+1$, if A is an interval on which h is increasing, -1 if A is an interval on which h is decreasing and 0 if A is a critical point.

1.2 Itineraries. Partial Order

The *itinerary* $\Im(x) = (A_0(x), A_1(x), \ldots)$ of a point $x \in I$ under h is a sequence of symbols in $\mathcal{A} = \{H_0, \ldots, H_m\} \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$, where

$$\begin{cases} A_k(x) = H_j, & \text{if } f^{\circ k}(x) \in H_j, \\ A_k(x) = \mathbf{c}_j, & \text{if } f^{\circ k}(x) = \mathbf{c}_j. \end{cases}$$

Not all sequences of appropriate symbols are in general admissible for a fixed m-modal map. There are rigorous criteria to test whether or not a such sequence is achieved as the itinerary of some point under a given map (see [5, 11]).

On the space of all admissible m-modal itineraries we put a partial order as follows. Given two distinct itineraries $\Im_1 = (A_1, A_2, \ldots)$ and $\Im_2 = (B_1, B_2, \ldots)$, there exists a smallest index $k \geq 1$ for which $A_k \neq B_k$, which we will call their *discrepancy*. We say by definition that $\Im_1 > \Im_2$ if

$$\left(\prod_{i=1}^{k-1} \operatorname{sign} A_i\right) A_k > \left(\prod_{i=1}^{k-1} \operatorname{sign} B_i\right) B_k.$$

We say, as usual, that $\Im_1 \geq \Im_2$ if either $\Im_1 > \Im_2$ or $\Im_1 = \Im_2$.

Remarks (1) This order is consistent with the order of points on the real line. If $\Im(x)$ and $\Im(x')$ are the itineraries of two different points x and x' under the same m-modal map h, then:

$$\Im(x) < \Im(x') \quad \Rightarrow \quad x < x',$$
$$x < x' \quad \Rightarrow \quad \Im(x) \leq \Im(x').$$

(2) We will say that two itineraries $\Im_1$ and $\Im_2$ are *comparable* if either $\Im_1 \geq \Im_2$ or $\Im_1 \leq \Im_2$. Two itineraries that belong to two different maps may not be comparable. This happens if and only if they both contain a critical point c_i in the position right before the discrepancy, and if in addition the itineraries of c_i under the corresponding two maps are distinct.

1.3 Kneading Data. Partial Order

The *kneading sequences* of the map h are the itineraries of its folding values:

$$\mathcal{K}_j = \mathcal{K}(\mathbf{c}_j) = \Im(f(\mathbf{c}_j)), \quad 41 \leq j \leq m - 1.$$

The *kneading-data* $\mathbf{K}$ of h is the m-tuple of kneading-sequences:

$$\mathbf{K} = (\mathcal{K}_1, \ldots, \mathcal{K}_m).$$

Given two m-modal maps f and g, we say that $\mathbf{K}(f) \ll \mathbf{K}(g)$ if $\mathcal{K}_j(f) \leq \mathcal{K}_j(g)$ whenever $\mathbf{c}_j$ is a local maximum and $\mathcal{K}_j(f) \geq \mathcal{K}_j(g)$ whenever $\mathbf{c}_j$ is a local minimum. Clearly, $\mathbf{K}(f)$ and $\mathbf{K}(g)$ may fail to be comparable, even if the individual kneading sequences of f are comparable to the corresponding ones of g. Therefore, "$\ll$" is only a partial order on the kneading-data achievable by m-modal maps.

1.4 Topological Entropy

The topological entropy of a map measures in some sense the complexity of the corresponding dynamical system, by counting how many "very different" orbits of arbitrarily large length that the map can produce. It was introduced in the sixties as a quantity that is invariant under continuous changes of coordinates (see [1]). It has better continuity properties than other measures of chaos (such as the Lyapunov exponents). Its uses, especially to determine the degree of chaos in a system, are an incentive for developing sustainable algorithms to compute it in general or for particular cases.

We will adopt here the original definition for the entropy of a continuous self-map of a compact topological space.

Definition 1.1 Consider X a compact topological space, $\mathcal{C}$ an open cover of X and $f: X \to X$ a continuous map. For each $k \in \mathbb{N}$ construct a new open cover:

$$\mathcal{C}^k(f) = \{C_0 \cap f^{-1}C_1 \cap \cdots \cap f^{-(k-1)}C_{k-1}, C_i \in \mathcal{C}\}.$$

We define the topological entropy of the map f on X with the cover $\mathcal{C}$ as:

$$h(f, \mathcal{C}) = \lim_{k \to \infty} \frac{1}{k} \log n(\mathcal{C}^k(f)) \in [0, \infty]$$

where $n(\mathcal{C}^k(f))$ denotes the cardinality of the smallest subcover of $\mathcal{C}^k(f)$. We define the entropy of f as:

$$h(f) = \sup\{h(f, \mathcal{C})/\mathcal{C} \text{ open cover}\}.$$

As both measures of a map's complexity, the topological entropy and the kneading-data are related as follows:

Theorem (See [14]) *The topological entropy for an m-modal map is determined by its kneading-data. Moreover, if $\mathbf{K}(f) \gg \mathbf{K}(g)$, then $h(f) \geq h(g)$.*

Theorem (See [14]) *Topological entropy depends continuously on kneading-data.*

2 Is Entropy Computable?

Given an explicit dynamical system, is it possible in principle to compute the associated topological entropy?

Unfortunately, the answer is in general: **NO**. It is known that certain cellular automaton self-maps of Cantor sets have topological entropy that is not algorithmically computable [8].

Let's suppress our ambition and rephrase the question. Is it even possible that the entropy is computable within a given error margin ϵ? Are there any particular families of maps for which this happens? And if so, how costly is it in terms of time and other programming resources?

It turns out that these are not a trivial questions to ask, either. For smooth 2-dimensional maps and for smooth diffeomorphisms of dimension ≥ 3, the topological entropy does not always depend continuously on parameters (see for example [12]), making computation difficult. However, the situation is not that unfortunate for one-dimensional maps, where we do possess strong continuity results which may help us in such ventures.

Theorem (See [2] or [17]) *The topological entropy function is continuous*:

$$h: \mathcal{C}^{\infty}(I, I) \to [0, \infty).$$

Corollary *For any d, the topological entropy function is continuous on the finite dimensional compact space consisting of all polynomial maps of the interval with degree $\leq d$.*

In addition, we have a few interesting alternative definitions for the topological entropy of m-modal maps:

A. If f is a piecewise monotone map of the interval, then

$$h(f) = \lim_{k\to\infty} \frac{1}{k} \log l(f^{\circ k}) = \inf_{k>0} \frac{1}{k} \log l(f^{\circ k})$$

where $l(f^{\circ k})$ is the lap number of $f^{\circ k}$ (see for example [13]).

B. If f strictly piecewise monotone, then

$$h(f) = \lim_{k\to\infty} \frac{1}{k} \log(\mathrm{Adm}(f, k))$$

where $\mathrm{Adm}(f, k)$ is the number of acritical admissible sequences of length k (see [14]).

C. If f has at most finitely many nonrepelling orbits, then

$$h(f) = \limsup_{k\to\infty} \frac{1}{k} \log^{+} \mathrm{Neg}(f^{\circ k})$$

where $\mathrm{Neg}(F)$ is the number of fixed points of F of negative type (see [15]).

To the best of the author's knowledge, none of them turned out to be computationally optimal. There are, however, different approaches to the case of m-modal maps. Reference [4] presents an approximation algorithm using Markov partitions that converges fairly rapidly and provides upper and lower bounds. It applies to piecewise monotonic interval maps, but can be adapted to other types. The method is flexible in treating multiple turning points.

Although the ideas presented here could be applied in a more general context, in this paper we will fix our attention on a subfamily of polynomial maps on the unit interval, more precisely the family of quartic polynomials that are compositions of two logistic maps: $q_\mu \circ q_\lambda$ where $q_\lambda: I \to I$ given by $q_\lambda(x) = \lambda x(1-x)$, $\forall \lambda \in [0, 4]$.

Some theoretical results are already known in the parameter space P^Q of this family. Although we do not have any of the classical monotonicity properties known within the logistic family (the entropy does not directly increase in P^Q with either parameter λ or μ), we do have a basic topological result [16]: The level-sets of the entropy, called isentropes, are connected subsets of P^Q. They seem to be either filled regions in P^Q, or curves connecting two boundary points, with strange shapes and visible singularities. At this stage, it is not clear which values of the entropy produce one-dimensional isentropes and which regions, and we do not know where the singularities occur. While developing a sustainable algorithm that estimates entropy in P^Q, this paper will produce a picture with a mathematically rigorous basis that displays all these phenomena in a trustable way.

The algorithm we present follows closely an idea used by [10] for maps with three monotone pieces, and is based on the intimate relationship between the topological entropy of a map and its kneading-data described in Sect. 1.

3 More on Kneading Data

Due to their symmetry, the maps $q_\mu \circ q_\lambda$ have only two significant kneading sequences: $\mathcal{K}_1 = \mathcal{K}_3$ and $\mathcal{K}_2$. This section contains a few remarks concerning the partial order on itineraries and the partial order on kneading data of maps with such symmetry.

Definition 3.1 Consider two copies I_1 and I_2 of the unit interval and two unimodal maps $f_1 \colon I_1 \to I_2$ and $f_2 \colon I_2 \to I_1$ with critical points $\gamma_1 \in I_1$ and $\gamma_2 \in I_2$, respectively. We call a *diorbit* under the pair (f_1, f_2) a sequence:

$$x \to f_1(x) \to f_2(f_1(x)) \to f_1(f_2(f_1(x))) \dots.$$

We say a diorbit is *critical* if it contains either critical point γ_1 or γ_2. A critical diorbit that contains both γ_1 and γ_2 will be called *bicritical*.

We call the *d'itinerary* of a point x under (f_1, f_2) the infinite sequence $\Im(x) = \{J_k(x)\}_{k \geq 0}$ of alternating symbols in $\{L_1, \Gamma_1, R_1\}$ and $\{L_2, \Gamma_2, R_2\}$ that expresses the positions (left, critical or right) of the iterates of x in I_1 and I_2 with respect to γ_1 or γ_2 (see [16]).

We define a partial order on d'itineraries in a similar way as we did for regular itineraries. The following lemma leads to a relationship between the critical d'itineraries under (f_1, f_2) and the kneading-data of $f_2 \circ f_1$.

Lemma 3.2 *Consider f_1 and f_2 as above and call*:

$$\Im(f_1, f_2)(\gamma_2) = (\Gamma_2, J_1, J_2, \dots, J_{2k}, J_{2k+1}, \dots)$$

the d'itinerary of γ_2 under (f_1, f_2) and $\mathcal{K}_1 = (A_1, A_2, \dots, A_k, \dots)$ the sequence in $\mathcal{A} = \{H_0, c_1, H_1, c_2, H_2, c_3, H_3\}$ that represents the first kneading-sequence of $f_2 \circ f_1$. Then, for any $k \geq 1$:

$$\begin{array}{lll}
A_k = H_0 & \textit{iff} & (J_{2k-1}, J_{2k}) = (L_1, L_2), \\
A_k = H_1 & \textit{iff} & (J_{2k-1}, J_{2k}) = (L_1, R_2), \\
A_k = H_2 & \textit{iff} & (J_{2k-1}, J_{2k}) = (R_1, R_2), \\
A_k = H_3 & \textit{iff} & (J_{2k-1}, J_{2k}) = (R_1, L_2), \\
A_k = c_1 & \textit{iff} & (J_{2k-1}, J_{2k}) = (L_1, \Gamma_2), \\
A_k = c_3 & \textit{iff} & (J_{2k-1}, J_{2k}) = (R_1, \Gamma_2), \\
A_k = c_2 & \textit{iff} & J_{2k-1} = \Gamma_1.
\end{array}$$

Proof The proof is an easy exercise. □

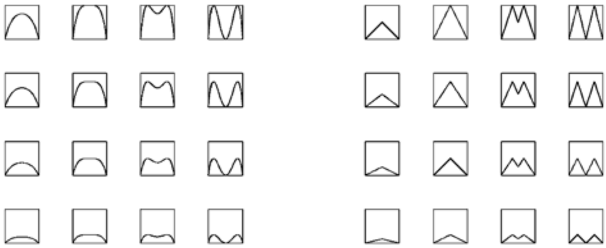


Fig. 1 The behavior of the quartic maps $q_\mu \circ q_\lambda$ for various values of the parameters λ and μ, showed by comparison with the behavior of the sawtooth family $T_b \circ T_a$ for various values of a and b

Lemma 3.3 *If two distinct pairs of unimodal maps* (f_1, f_2) *and* (f_1', f_2') *have d'itineraries of* γ_1 *and* γ_2 *such that* $\Im((f_1, f_2))(\gamma_1) \leq \Im((f_1', f_2'))(\gamma_1)$ *and* $\Im((f_1, f_2))(\gamma_2) \leq \Im((f_1', f_2'))(\gamma_2)$, *then their kneading data* $\mathbf{K}(f_2 \circ f_1) \ll \mathbf{K}(f_2' \circ f_1')$.

Proof The result follows easily from Lemma 3.2 and the definition of the order on d'itineraries and on kneading-data. □

When then are the kneading-data of two maps $f_2 \circ f_1$ and $f_2' \circ f_1'$ comparable? A first concern is that one of the critical d'itineraries under (f_1, f_2) may not be comparable with the corresponding d'itineraries under (f_1', f_2'). This does not happen very often. $\Im(f_1, f_2)(\gamma_1)$ is not comparable with $\Im(f_1', f_2')(\gamma_1)$ if and only if (1) they have Γ_2 on the last common position, and (2) the d'itineraries of γ_2 are not equal to each other. According to Lemma 3.3, comparable kneading-data for $f_2 \circ f_1$ and $f_2' \circ f_1'$ means comparable d'itineraries that satisfy one of the following conditions:

(1) $\Im((f_1, f_2))(\gamma_1) \leq \Im((f_1', f_2'))(\gamma_1)$ and $\Im((f_1, f_2))(\gamma_2) \leq \Im((f_1', f_2'))(\gamma_2)$, in which case $\mathbf{K}(f_2 \circ f_1) \leq \mathbf{K}(f_2' \circ f_1')$ or
(2) $\Im((f_1, f_2))(\gamma_1) \geq \Im((f_1', f_2'))(\gamma_1)$ and $\Im((f_1, f_2))(\gamma_2) \geq \Im((f_1', f_2'))(\gamma_2)$, in which case $\mathbf{K}(f_2 \circ f_1) \geq \mathbf{K}(f_2' \circ f_1')$.

These are the conditions we strive to obtain in search of maps with comparable kneading-data.

4 Why P^T, no More and no Less?

To help us calculate the entropy for maps with parameters in P^Q, we use the model space P^T of pairs of tent maps (Fig. 1). This method of comparing a space of polynomials with a simpler space of piecewise linear maps is not new. For topological entropy-related results in P^Q, we preferred the model space P^{ST} of stunted sawtooth maps (see [14]), which preserves homeomorphically the significant topological structures in P^Q. For computational purposes, however, using tent maps is not only sufficient, but also more helpful, as the following sections will show.

Recall that for $0 \leq a \leq 2$, we define the tent map $T_a\colon I \to I$ as: $T_a(x) = a(1 - |x - 1|)$. For a pair $(a, b) \in [0, 2]^2$, the composition $T_b \circ T_a$ will be either a 3-modal or a one-modal sawtooth map with slope $\pm ab$. What makes this model attractive is the trivial behavior of this family with respect to the entropy function. It is known that the entropy of any sawtooth map with slope $\pm s$ is equal to $\log s$, if $s \geq 1$ and equal to zero otherwise (see [2] for proof and details). To simplify, we consider then the subset $\{(a, b)$ such that $ab \geq 1\} \subset P^{ST}$ and we reparametrize it by $s = \log a$, $t = \log b$:

$$U^T = \{(s, t) \in [-\log 2, \log 2]^2, s + t \geq 0\}$$

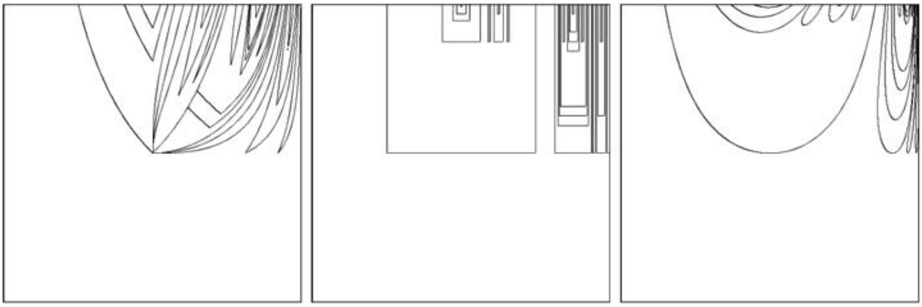


Fig. 2 Left bones of order $n = 4$ in P^T, P^{ST}, P^Q (see [14]). The left bones of order n are defined as the subsets of parameters for which the left critical point $\gamma_1 = \frac{1}{2} \in I_1$ returns to itself after $2n$ alternate iterations. Compare with the isentropes pictures in the three parameter spaces in Sect. 7

U^T encompasses all entropy values in $[0, \log 4]$, and does it in a very tidy way, as shown below. As there is no danger of confusion, the maps in U^T will also be denoted by T_s, but with $s \in [-\log 2, \log 2]$.

For a fixed value $h^* \in [0, \log 4]$, we call the h^*-isentrope in U^T the subset of parameters where the entropy of the corresponding sawtooth function is h^*:

$$i^T(h^*) = \{(s, t) \text{ with } h(T_t \circ T_s) = h^*\}.$$

Remark For any $h^* \in [0, \log 4]$, the corresponding isentrope is a line segment of slope -1:

$$i^T(h^*) = \{(s, t) \in U^T, s + t = h^*\}$$

with the two boundary points: $(s_l, t_l) = (h^* - \log 2, \log 2)$ and $(s_r, t_r) = (\log 2, h^* - \log 2)$.

U^T represents all possible entropy values in $[0, \log 4]$. We do not know if it also encompasses all possible kneading-data of symmetric $(+, -, +, -)$ 3-modal maps. However, as we will see, all we want is to find an element in U^T with kneading-data *comparable* with any $\mathbf{K}$ attained in P^Q. There will be no homeomorphism between the combinatoric structure in P^Q and P^T, unlike between P^Q and P^{ST}. Observe, for instance, the topological differences between the computer-generated pictures in Fig. 2, showing for each space some of the algebraic curves of parameters where the critical point γ_1 is periodic.

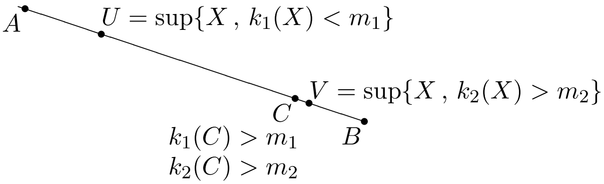
5 The Search for Kneading-Data

We say that a boundary-anchored m-modal map of the interval is *critically preperiodic* or *Markov* if all critical points have finite orbits. We will prove in this section the following:

Theorem 5.1 *Consider an isentrope $i^T(h^*) \subset U^T$ which does not contain any Markov maps. Then, for any arbitrary map in g in P^Q, there exists a map f along $i^T(h^*)$ with kneading-data comparable to $\mathbf{K}(g)$.*

As shown in Sect. 4, for any fixed $h^* \in U^T$, the h^*-isentrope $i^T(h^*) \subset U^T$ is the line segment $s + t = h$ with boundary points $L = (s_l, t_l) = (h^* - \log 2, \log 2)$ and $R = (s_r, t_r) = (\log 2, h^* - \log 2)$. For convenience, we introduce the following notations and conventions.

Fig. 3 Portion of a bone that illustrates the positions of U and V where the combinatorics change



We put an order on $\overline{LR} = i^T(h^*)$ from L to R (i.e. $X < Y$ if $d(X, L) < d(Y, L)$. We say as usual that $X \le Y$ if $d(X, L) \le d(Y, L)$). The d'itineraries at points $(s, t) \in i^T(h^*)$ are called (k_1, k_2). At a particular point $X \in i^T(h^*)$, the corresponding sawtooth map is f_X and the pair of critical d'itineraries is $(k_1(X), k_2(X))$.

The notation $k_1 \triangle m_1$ and $\mathbf{K}(f)\triangle\mathbf{K}(g)$ stands for "the two d'itineraries or kneading data are comparable," and $k_1 \blacktriangle m_1$ and $\mathbf{K}(f)\blacktriangle\mathbf{K}(g)$ for "not comparable."

Lemma 5.2 *Fix a map $g \in P^Q$ and an isentrope $i^T(h^*) \subset U^T$ which does not contain Markov maps. Call (m_1, m_2) the pair of d'itineraries of g. Suppose that there are two points $A, B \in i^T(h^*)$ such that*

$$\textit{either} \begin{cases} k_1(A) \le m_1, & k_2(A) \ge m_2, \\ k_1(B) \ge m_1, & k_2(B) \le m_2 \end{cases} \quad \textit{or} \begin{cases} k_1(A) \ge m_1, & k_2(A) \le m_2, \\ k_1(B) \le m_1, & k_2(B) \ge m_2. \end{cases}$$

Then there exists a point $C \in i^T(h^*)$ between A and B such that:

$$\textit{either} \begin{cases} k_1(C) \le m_1, \\ k_2(C) \le m_2 \end{cases} \quad \textit{or} \begin{cases} k_1(C) \ge m_1, \\ k_2(C) \ge m_2. \end{cases}$$

In other words, there exists a point C such that the corresponding map f_C has kneading-data comparable with $\mathbf{K}(g)$.

Proof As we know that there are no Markov maps on $i^T(h^*)$, it follows that there may be points along $\overline{AB}$ where either $k_1 \blacktriangle m_1$ or $k_2 \blacktriangle m_2$, but not both simultaneously (Fig. 3).

Notice first that if at some point X we have $k_1(X)\blacktriangle m_1$, then either $k_2(X) < m_2$ or $k_2(X) > m_2$, otherwise the map f_X would be Markov. Similarly, $k_2(X)\blacktriangle m_2$ implies $k_1 < m_1$ or $k_1 > m_1$. So, if we find a point C with $k_1(C) = m_1$ or $k_2(C) = m_2$, we are done.

We can assume WLOG that $k_1(A) < m_1, k_2(A) > m_2$ and $k_1(B) > m_1, k_2(B) < m_2$. Call $U = \sup\{X \in \overline{AB}$ with $k_1(X) < m_1\}$. Then either $k_1(U) = m_1$ or $k_1(U)\blacktriangle m_1$ (because both $k_1 < m_1$ and $k_1 > m_1$ are open conditions). Similarly, call $V = \sup\{X \in \overline{AB}$ with $k_2(X) > m_2\}$, so either $k_2(V) = m_2$ or $k_2(V)\blacktriangle m_2$.

Clearly $U \neq V$, otherwise $f_U = f_V$ would be Markov. Assume WLOG that $U < V$. This implies $k_1(V) > m_1$, otherwise $k_1(V) = m_1$ or $k_1(V)\blacktriangle m_1$ and f_V would be Markov. If $k_2(V) = m_2$, then we take $C = V$ and we are done. If $k_2(V)\blacktriangle m_2$, then any small neighbourhood $\mathcal{V} \ni V$ contains at least one point C such that $k_2(C) > m_2$ (by the definition of sup) and $k_1(C) > m_1$ (because the ">" condition is open) and we are again done. □

Proof of Theorem 5.1 Let us notice that the first d'itinerary k_1 assumes the absolute maximum $k_1^{\max} = (R_2 L_1 L_2 \ldots)$ at the left end-point L. The second itinerary $k_2(L)$ can't be finite, or else the map corresponding to L would be Markov. So we have either $k_2 \le m_2$ or $k_2 \ge m_2$. Similarly, the second d'itinerary assumes its maximum $k_2^{\max} = (R_1 L_2 L_1 \ldots)$ at R, and also

$k_1(R) \le m_1$ or $k_1(R) \ge m_1$. If either (1) $k_1 \ge m_1$ and $k_2 \ge m_2$ or (2) $k_1 \le m_1$ or $k_2 \le m_2$, then we are done. If not, then by Lemma 5.2 there is an appropriate point $C \in i^T(h^*)$ between A and B, and we are also done. □

6 The Idea of the Algorithm

We fix a parameter $(\lambda, \mu) \in P^Q$ and aim to estimate the entropy $h(g)$ of the corresponding map $g = q_\mu \circ q_\lambda$. We start by computing its pair of critical d'itineraries, which we will use to perform an interesting algorithmic search.

Suppose we have an underestimate h_0 and an overestimate h_1 for $h(g)$. To start the algorithm, set the bounds as the a priori values $h_0 = 0$ and $h_1 = \log 4.1$ (we chose 4.1 and not 4 for reasons that will become clear later). We want to improve our two estimates within an error range of, say $|h_1 - h_0| < \epsilon = 10^{-4}$; when this error is reached, the algorithm stops and returns $h_0(g) = h_0$ as the final underestimate and $h_1(g) = h_1$ as the overestimate. We use an iterated bisecting technique. At each step, consider the average $h^* = \frac{1}{2}(h_0 + h_1)$. We search along the isentrope $i^T(h^*) = \{(s,t), h(T_t \circ T_s) = h^*\}$ for a map $f = T_t \circ T_s$ that has kneading-data comparable with $\mathbf{K}(g)$. If $\mathbf{K}(f) \gg \mathbf{K}(g)$, then $h^* = h(f) \ge h(g)$ gives us a better upper-estimate than h_1, in which case we reassign this value to h_1. If $\mathbf{K}(f) \gg \mathbf{K}(g)$, then $h = h(f) \le h(g)$ is a better lower-estimate than h_0, so we reassign this value to h_0. We continue to iterate until $|h_1 - h_0| < 10^{-4}$, when we would have achieved the level of approximation needed.

Theorem 5.1 assures us that there will always be such a map $f = T_t \circ T_s$ with $\mathbf{K}(f)$ comparable to $\mathbf{K}(g)$, provided none of the isentropes picked by the algorithm contains Markov maps. So let's see if our isentropes are indeed free of Markov maps.

Let $f : I \to I$ be a boundary anchored m-modal Markov map. If $P = [x_0, x_1, \ldots, x_m]$ is the ordered union of its finite critical orbits, then $I \backslash P$ is a finite union of open intervals, whose closures $J_k = (x_{k-1}, x_k)$ form a *Markov partition* of I. We define the *Markov matrix* $M = (M_{ij})$ of f as follows: For each pair $(i, j) \in \{1, \ldots, p\}^2$, we set $M_{ij} = 1$ if $f(J_i)$ covers J_j, and we set $M_{ij} = 0$ if $f(J_i) \cap J_j = \Phi$.

We recall two known results related to Markov matrices that justify our algorithm. (For a nice approach to computing the entropy using Markov partitions, see [4].)

Theorem *The topological entropy of a Markov map is* $\log \lambda$ *where* λ *is the maximal eigenvalue of its Markov matrix* (*see for example* [7]).

Theorem *The associated Markov matrix* M *of an* m*-modal function* f *is invertible*, *so that its largest eigenvalue* λ *is an algebraic unit.*

Proof [1] As before, let $x_0 < \cdots < x_m$ be the points of P. Then we can write $f(x_k) = x_{\sigma(k)}$ where σ is a permutation of $\{0, 1, \ldots, m\}$. If J_k is the interval $[x_{k-1}, x_k]$ then the permutation σ gives rise to an $m \times m$ Markov matrix M with entries M_{ij} equal to one or zero, according to whether $f(J_i)$ does or does not cover J_j. We must prove that M is invertible.

Let A be the $(m+1) \times (m+1)$ permutation matrix with entries A_{ij} equal to one if $\sigma(i) = j$ and zero otherwise. Then M can be constructed from A in three steps:

(1) In each row of A, replace every entry to the left of a one by a one.

[1] Proof communicated by Professor John Milnor, unpublished work, 12/01/2004.

(2) For each $k > 0$, replace the row R_k of A by $\pm(R_k - R_{k-1})$, choosing the sign so that the entries are non-negative.
(3) Throw away the 0-th row and column of the resulting matrix. This will be the required M.

It is not hard to see that each of these matrix modifications preserves the determinant, up to sign. Since A has determinant ± 1, it follows that M has determinant ± 1. $\square$

Corollary 6.1 *The topological entropy of a Markov map is the logarithm of an algebraic unit.*

With the corollary above, proving that our isentropes contain no Markov maps is now an easy task. Indeed, it is not hard to see that all entropy levels h^* chosen for the isentropes are of the form $h^* = \frac{N}{2^n}\log 4.1$, where N and n are positive integers. Suppose such an h^*-isentrope contained a Markov map. Then, by the corollary, it would follow that $(4.1)^{\frac{N}{2^n}}$ is an algebraic unit, contradiction.

Even knowing that along each such fixed h^*-isentrope there is a map f with kneading-data $\mathbf{K}(f)$ comparable to $\mathbf{K}(g)$, we still have to construct an effective algorithm that searches for it. We start by checking the endpoints $L = (s_l, t_l) = (h^* - \log 2, \log 2)$ and $R = (s_r, t_r) = (\log 2, h^* - \log 2)$. If $k_2(L) \geq m_2$, then $\mathbf{K}(f_L) \geq \mathbf{K}(g)$ and we are done. Symmetrically, if $k_1(R) \geq m_1$, we have $\mathbf{K}(f_R) \geq \mathbf{K}(g)$ and we are done. Given that the h^*-isentrope has no Markov maps, the only other possible case is:

$$\begin{cases} k_1(L) \geq m_1, \\ k_2(L) \leq m_2 \end{cases} \quad \text{and} \quad \begin{cases} k_1(R) \leq m_1, \\ k_2(R) \geq m_2. \end{cases}$$

By Theorem 5.1, we know that there is a map f along $\overline{LR}$ with kneading-data comparable to $\mathbf{K}(g)$. We consider the midpoint $M = (s_m, t_m) = (\frac{h}{2}, \frac{h}{2})$. If $k_1(M) \geq m_1$ and $k_2(M) \geq m_2$, then we have $\mathbf{K}(M) \geq \mathbf{K}(g)$, so we have successfully found $f = f_M$. If $k_1(M) \leq m_1$ and $k_2(M) \leq m_2$, then we have found $f = f_M$ with $\mathbf{K}(M) \leq \mathbf{K}(g)$. If $k_1(M) \geq m_1$ and $k_2(M) \leq m_2$, then we replace $L = (s_l, t_l) = (s_m, t_m)$. If $k_1(M) \leq m_1$ and $k_2(M) \geq m_2$, then we replace $R = (s_r, t_r) = (s_m, t_m)$. In both cases, Theorem 5.1 grants us the existence of an f within the new interval $\overline{LR}$ with $\mathbf{K}(f)$ comparable to $\mathbf{K}(g)$. We iterate the algorithm with the new end points $L = (s_l, t_l)$ and $R = (s_r, t_r)$. We construct a nested sequence of intervals around an f with the required properties. Note (from the proof of Lemma 5.2) that in most cases any point in a small neighborhood of f satisfies, so it is likely that in a fairly short computation time (i.e. small number of iterations) we find an answer along the h^*-isentrope.

This plan could have one main problem: we do not know yet that all midpoints obtained by this iterated bisection have critical d'itineraries comparable to m_1 and m_2, respectively.

Note first that all midpoints produced by the algorithm have coordinates of the form:

$$(s, t) = \left(\frac{p}{2^{r+1}}h + \frac{q}{2^r}\log 2, \frac{p'}{2^{r'+1}}h + \frac{q'}{2^{r'}}\log 2\right)$$

where p, p', q, q' are positive odd integers and r, r' are positive integers.

In other words, we are only picking pairs of tent maps with slopes of the form:

$$a = \exp(s) = 2^{\frac{q}{2^r}} H^{\frac{p}{2^{r+1}}} \quad \text{and} \quad b = \exp(t) = 2^{\frac{q'}{2^{r'}}} H^{\frac{p'}{2^{r'+1}}}$$

where $H = \exp(h)$. We will show next that the critical d'itineraries for all such pairs (a, b) cannot contain critical points.

Suppose the opposite: the algorithm produces a pair (a, b) with $ab = \exp(s)\exp(t) = \exp(s + t) = H$, such that the critical point $\frac{1}{2}$ returns to itself after a finite number of alternate iterates of $T_a(x) = a(1 - |x - 1|)$ and $T_b(x) = b(1 - |x - 1|)$. Without writing the terms explicitly, this condition translates as an equation having the general form: $aP(H) + HQ(H) = 1$ (or $bP'(H) + HQ'(H) = 1$, which could be analyzed similarly). Here $H = ab = \exp(h)$ and P and Q are polynomials with coefficients only 1 and 2.

Substituting $a = 2^{\frac{q}{2^r}} H^{\frac{p}{2^{r+1}}}$, we have:

$$2^{\frac{q}{2^r}} H^{\frac{p}{2^{r+1}}} P(H) + HQ(H) = 1.$$

In other words:

$$2^{2q} H^p P^{2^{r+1}}(H) = [1 - HQ(H)]^{2^{r+1}}. \tag{1}$$

As 41 is a prime number, we have the valuation function of $v_{41}\colon \mathbb{Q} \to \mathbb{N}$ given by $v_{41}(\frac{a}{b}) = m$ if $\frac{a}{b} = 41^m \frac{c}{d}$ such that c and d are not divisible by 41 (see [9]). We consider the extension of this function to the totally ramified field extension $\mathbb{Q}_{41}(4.1^{\frac{1}{2^n}})$, which we call V_{41} (see [3] for definitions and details).

To start with, we have that:

$$V_{41}(4.1^{\frac{1}{2^n}}) = 1$$

so

$$V_{41}(H) = N V_{41}(4.1^{\frac{1}{2^n}}) = N.$$

It follows that $V_{41}(P(H))$ is an integer B.

We valuate the left side of (1):

$$\begin{aligned} V_{41}(2^{2q} H^p P^{2^{r+1}}(H)) &= V_{41}(2^{2q}) + V_{41}(H^p) + V_{41}(P^{2^{r+1}}(H)) \\ &= 0 + pV_{41}(H) + 2^{r+1} V_{41}(H) = pN + 2^{r+1} B. \end{aligned}$$

We valuate the right side of (1):

$$V_{41}([1 - HQ(H)]^{2^{r+1}}) = 2^{r+1} V_{41}(1 - HQ(H)) = 0.$$

In conclusion:

$$pN + 2^{r+1} B = 0$$

which is a contradiction, as the left side is an odd integer.[2]

We have proved that neither one of the critical d'itineraries at (a, b) contains critical points. It automatically follows that they are comparable to any d'itineraries, in particular to m_1 and m_2, respectively.

Remark This proves that the points generated by our search algorithm never fall on bones or capture components (see [16] for more on bones and combinatorics).

[2] I am indebted to Professor David Grant for useful conversations.

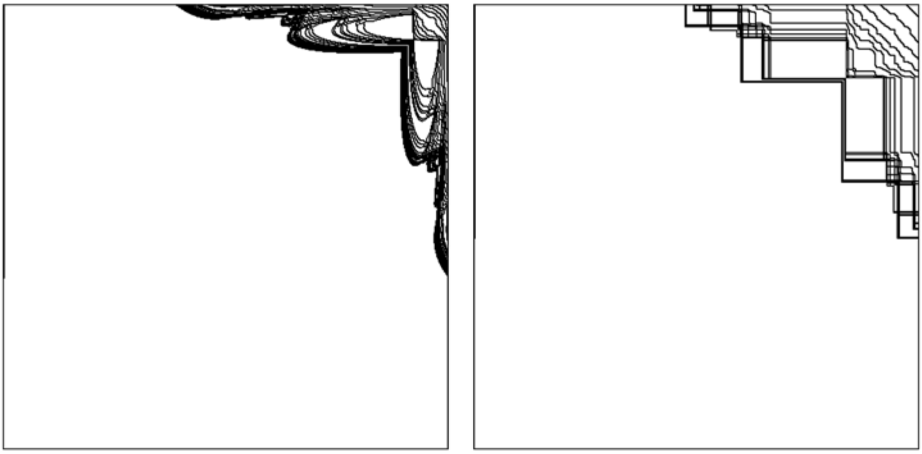


Fig. 4 Some of the isentropes in P^Q (*left*) and P^{ST} (*right*) generated by a C program following the algorithm discussed in Sects. 5 and 6. To give some clarity to the pictures, the closer we are to the upper right corner, the fewer of the actual isentropes are shown

7 The Final Picture and Some Comments

The picture of isentropes in P^Q shown in Fig. 4 is obtained by calculating the entropy for a grid of 480×480 parameter points in $[0, 4]^2$. For a complete version of the C program that implements the algorithm described in the paper, see the following reference: www.amath.colorado.edu/faculty/radulescu.

Some isentropes are easily identifiable. Clearly, the upper right corner, with coordinates $(\lambda, \mu) = (4, 4)$, is the isentrope corresponding to $h^* = \log 4$. The other isentropes are either curves or filled regions. A little investigation shows that the region that contains the two largest symmetric windows is the $\log 2$-isentrope. Most curves that constitute arc-isentropes or separate region-isentropes exhibit lots of singularities. An interesting result in this sense is offered by [6], which discusses the combinatorial and smoothness properties of the boundary of chaos (i.e., boundary of the 0-isentrope), for both parameter spaces of stunted sawtooth maps and polynomials.

To end, I will point out the potential problems that one might expect when running the program.

(1) The program does not generate infinite d'itineraries; it only works with truncations to the first N iterates (for the picture above, we took $N = 40$). Therefore, the algorithm may find two sequences to be equal when in fact they aren't, except that the discrepancy occurs after the Nth position. This does not raise serious concerns: if we work with reasonably long truncated d'itineraries we still get reasonable estimates. Besides, dealing with infinity is intrinsically problematic for any computer program, and the theoretical algorithm explicitly requires such concepts. So this problem can't be helped.
(2) When creating the d'itineraries, the double or long precision may fail to detect if one iterate hits the critical point or just gets very close to it. However, this does not seem to happen for the set-up we chose.
(3) The search along an isentrope in P^T could require too many bisections, that would cause the program to stall for too long on that isentrope. This situation does not seem to occur in practice, either.

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